JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 2, May 2012

A FAST LAGRANGE METHOD FOR LARGE-SCALE IMAGE RESTORATION PROBLEMS WITH REFLECTIVE BOUNDARY CONDITION

SeYoung Oh* and SunJoo Kwon**

Abstract. The goal of the image restoration is to find a good approximation of the original image for the degraded image, the blurring matrix, and the statistics of the noise vector given. Fast truncated Lagrange (FTL) method has been proposed by G. Landi as a image restoration method for large-scale ill-conditioned BTTB linear systems([3]). We implemented FTL method for the image restoration problem with reflective boundary condition which gives better reconstructions of the unknown, the true image.

1. Introduction

Obtaining an accurate model of image blurring essentially requires the identification of the blur operator which is called a point spread function(PSF) and the choice of appropriate boundary conditions. The former is related to the continuous infinite dimensional problem and which decides the essential structure of the involved system matrix. The latter has a substantial impact in the precision of the reconstruction especially close to the boundaries of the image(presence of ringing effects) since the observed image is always finite([1]).

For getting the useful approximation of the true image, general linear, discrete ill-posed problems of the image restoration arisen from a first-kind Fredholm integral equations has to be replaced by a Tikhonov

Received April 23, 2012; Accepted May 07, 2012.

2010 Mathematics Subject Classification: Primary 65F22, 65K10, 65F10.

Key words and phrases: fast truncated Lagrange method, image restoration, reflective boundary condition, constrained linear least-square problems, discrete cosine transformation.

Correspondence should be addressed to SunJoo Kwon, sjkw@cnu.ac.kr.

This study was financially supported by research fund of Chungnam National University in 2010.

regularization

$$\min_x \|\mathcal{H}x - y\|_2^2 + \gamma \|x\|_2^2, \tag{1.1}$$

where $y \in R^{n^2}$ represents the observed image and $x \in R^{n^2}$ is the true image, γ is a positive regularization parameter. Note that the blurring matrix $\mathcal{H} \in R^{n^2 \times n^2}$ is a large scale ill-conditioned([8, 9]).

The aim of deblurring problem is to recover true image from blurred and noisy images for given space invariant point spread functions. Because of the blurring process, the boundary values of degraded image are not completely determined by the original image inside the scene. The choice of the most appropriate boundary condition is an important aspect in the modelization of image blurring. There are various studies about boundary condition in the image restoration. Zero and periodic boundary conditions bring an artificial discontinuity at the boundary, which implies a ringing effect of the reconstructed image. To reduce ringing effect, however, the use of reflective boundary condition was considered in ([6, 13]).

The fast truncated Lagrange(FTL) method is well suited for large-scale ill-conditioned BTTB linear systems which is from an image restoration problem with zero boundary condition ([3]). FTL method uses a quasi-Newton method applying to the first-order optimality conditions of the Lagrangian function for the equality constrained minimization of (1.1). Since the Hessian of the Lagrangian can be approximated by a BCCB(block circulant with circulant block) matrix, the two-dimensional fast Fourier transform(FFT) is used to compute the quasi-Newton step. The quasi-Newton iteration is terminated according to the discrepancy principle. The corresponding Lagrange multiplier of an iterative Lagrange method acts as a regularization parameter.

We analyzed the efficiency of FTL-method for the image restoration problem with the reflective boundary condition. The use of the reflective boundary condition results in a blurring matrix that is a Toeplitz-plus-Hankel matrix with Toeplitz-plus-Hankel blocks. Although these matrices have the complicated structures, they can always be diagonalized by the two dimensional discrete cosine transform matrix, provided that the blurring function is symmetric. Thus their inverses can be obtained by using fast cosine transform(FCT). Because an FCT requires only real multiplications and can be done at half of the cost of an FFT, inversion of these matrices is faster than that of those matrices obtained from classical zero or periodic boundary conditions([6]).

The outline of this paper is organized as follows : Section 2 recalls the FTL method in [3]. In Section 3, we discuss the implementation of FTL method for Toeplitz-plus-Hankel matrix with Toeplitz-plus-Hankel blocks linear system. Section 4 reports the efficiency of the reflective boundary condition in the FTL method. Finally, Section 5 is devoted to conclusions and options for the future research.

2. Fast truncated Lagrange method

This section briefly mentions the fast truncated Lagrange method suggested by G. Landi in [3]. This method for large scale image restoration problems with zero boundary condition is based on the Lagrange method and the discrepancy principle. It does not, however, require any prior good estimates of the regularization parameter.

Minimization problem (1.1) can be replaced by the equality constrained minimization,

$$\begin{aligned} &\min_x && \frac{1}{2}\|x\|^2 \\ &\text{subject to} && \frac{1}{2}\|\mathcal{H}x - y\|^2 = \epsilon, \end{aligned} \tag{2.1}$$

where ϵ is a small positive parameter, likely $0 < \epsilon \ll \frac{\delta^2}{2}$ if δ is the error norm. The Lagrangian function for (2.1) becomes

$$L(x, \lambda) = \frac{1}{2}\|x\|^2 + \lambda\left(\frac{1}{2}\|\mathcal{H}x - y\|^2 - \epsilon\right) \tag{2.2}$$

with the Lagrange multiplier λ. The Hessian of the Lagrangian is

$$\nabla^2_{xx} L(x, \lambda) = I + \lambda \mathcal{H}^*\mathcal{H}. \tag{2.3}$$

The structure of the matrix $\mathcal{H}^*\mathcal{H}$ is a block Toeplitz matrix with Toeplitz blocks(BTTB) under the zero boundary condition. It can be approximated by C^*C with a block circulant matrix with circulant blocks (BCCB) structure. A BCCB approximation matrix of the Hessian (2.3),

$$Q(\lambda) = I + \lambda C^*C,$$

is also symmetric and positive definite. Thus $Q(\lambda)$ can be easily inverted by using fast Fourier transformations(FFTs).

Since (2.1) is a convex problem, the first-order conditions are sufficient conditions for optimality. A quasi-Newton direction $(\Delta x^T, \Delta\lambda)$ is obtained as the solution of the nonlinear equations:

$$\begin{pmatrix} Q(\lambda) & \nabla_x h(x) \\ \nabla_x h(x)^T & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} \nabla_x L(x,\lambda) \\ h(x) \end{pmatrix}, \tag{2.4}$$

where $h(x) = \frac{1}{2}\|\mathcal{H}x - y\|^2 - \epsilon$. The explicit solution of (2.4) can be computed as

$$\begin{aligned} \Delta x &= -Q(\lambda)^{-1}(\nabla_x L(x,\lambda) + \nabla_x h(x)\Delta\lambda), \\ \Delta\lambda &= (\nabla_x h(x)^T Q(\lambda)^{-1}\nabla_x h(x))^{-1}\{h(x) \\ &\qquad - \nabla_x h(x)^T Q(\lambda)^{-1}\nabla_x L(x,\lambda)\}. \end{aligned} \tag{2.5}$$

For any $v \in R^{n^2}$, the product $Q(\lambda)^{-1}v$ can be computed by using FFTs. In the FTL algorithm, the next iterate is computed as

$$x_{new} = x + \alpha\Delta x, \lambda_{new} = \lambda + \alpha\Delta\lambda, \tag{2.6}$$

where the search direction $(\Delta x^T, \Delta\lambda)$ is the quasi-Newton direction (2.5) and the step-length α can be chosen by Armijo's condition. That is, letting the merit function $m(x,\lambda) = \frac{1}{2}\left\|\nabla_{(x,\lambda)} L(x,\lambda)\right\|^2$, the step length α is chosen by the first number of the sequence $\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^i}, \ldots\}$ which satisfies the Armijo's condition :

$$m(x+\alpha\Delta x, \lambda+\alpha\Delta\lambda) \le m(x,\lambda) + \mu\alpha(\Delta x^T, \Delta\lambda)\nabla_{(x,\lambda)} m(x,\lambda),$$

where $\mu = 10^{-4}$.

3. Reflective boundary condition

One image is shown only in a finite region and so points near the boundary of a blurred image are affected by information outside the field of view. Reflective boundary condition is that the pixels outside image is a mirror image of the scene inside the image borders. Periodic boundary condition imply that the image repeats itself endlessly in all directions. Zero boundary condition is that the pixels outside the borders of the image are all zero. Each boundary condition makes the PSF matrix into a different special structure ([12]).

This section focuses on the implementation of FTL method for the image restoration problem under the reflective boundary condition. Considering the reflective boundary condition, coefficient matrix $\mathcal{H}$ in (1.1) has a Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks structure. Let H be the $n \times n$ PSF matrix that define $\mathcal{H}$. The first column of Toeplitz-plus-Hankel matrix with Toeplitz-plus-Hankel blocks $\mathcal{H}$ can be found by applying *dctshift* to the extended matrix H_e made by embedding H in the middle and padding the zero matrix in the sides. The spectrum of $\mathcal{H}$ can be computed by using *dct2*(H_e) where *dct2* denotes the two-dimensional discrete cosine transformation ([10]). Let *vector* be the operator converting a $n \times n$ array in a n^2 vector and *extract*, the operator which extracts from a $2n \times 2n$ matrix the real part of its $n \times n$ central submatrix. Then the product $y = \mathcal{H}x$ is obtained as

$$y = vector(extract(idct2(dct2(H_e).*dct2(X_e)))),$$

where *idct2* is the inverse process of *dct2*. Here X_e is the $2n \times 2n$ central submatrix from extended array,

$$\begin{pmatrix} X_x & X_{ud} & X_x \\ X_{lr} & X & X_{lr} \\ X_x & X_{ud} & X_x \end{pmatrix},$$

where a matrix $X_{n\times n}$ is rearranged n^2-length vector x and $X_{lr} = fliplr(X)$, $X_{ud} = flipud(X)$, $X_x = fliplr(X_{ud})$.

The lemma below shows that for a symmetric blurring, the blurring matrix $\mathcal{H}$ can be diagonalized by the two-dimensional discrete cosine transform matrix. For the detail proof, see [6].

Lemma 3.1. *If the point spread function* $\mathtt{h}$ *is symmetric, then* $\mathcal{H}$ *can be diagonalized by the two-dimensional discrete cosine transform matrix* $C \otimes C$*, where the* (i, j)*th entry of* $n \times n$ *discrete cosine transform matrix* C *is given by*

$$[C]_{i,j} = \sqrt{\frac{2-\delta_{i1}}{n}} \cos\left(\frac{(i-1)(2j-1)\pi}{2n}\right), \quad 1 \le i, j \le n, \tag{3.1}$$

where δ_{ij} *is the Kronecker delta function.*

The solver for a deblurring problem with the reflective boundary condition is twice as fast as problem with other boundary conditions since fast cosine transform requires only real multiplications([4, 6]).

Let $\mathfrak{C} = \{(C \otimes C)^T \Lambda (C \otimes C) | \Lambda \text{ is an } n^2 \times n^2 \text{ real diagonal matrix}\}$. For nonsymmetric point spread functions, matrices in $\mathfrak{C}$ may be used as preconditioners to speed up the convergence of iterative methods. Given a matrix B, we define the optimal cosine transform preconditioners $c(B)$ to be the minimizer of $\|S - B\|_F$ over all S in $\mathfrak{C}$.

Lemma 3.2. ([7]) *Let* $\mathtt{h}$ *be an arbitrary point spread function and* $\mathcal{H}$ *be the blurring matrix of* $\mathtt{h}$ *with the reflective boundary condition imposed. Then the optimal cosine transform preconditioner* $c(\mathcal{H})$ *of*

$\mathcal{H}$ is the blurring matrix corresponding to the symmetric point spread function h *given by*

$$[\mathtt{h}_s]_{i,j} \equiv ([\mathtt{h}]_{i,j} + [\mathtt{h}]_{i,-j} + [\mathtt{h}]_{-i,j} + [\mathtt{h}]_{-i,-j})/4$$

with the reflective boundary condition imposed.

Considering the problem (2.1) with reflective boundary condition, the matrix $\mathcal{H}$ has the structure of Toeplitz-plus-Hankel matrix with Toeplitz-plus-Hankel blocks. The problem (2.1) is replaced to the unconstrained minimization problem of the Lagrange function $L(x, \lambda)$. Applying the first order optimality conditions for this problem, the following new problem is obtained,

$$\begin{aligned} &\min_{x,\lambda} \qquad L(x,\lambda) \\ &\text{subject to } \nabla_{x,\lambda} L(x,\lambda) = 0, \lambda \geq 0. \end{aligned} \tag{3.2}$$

THEOREM 3.3. *Under the reflective boundary condition, the iterative solution of problem (3.2) based on the FTL method is*

$$\begin{aligned} x_{k+1} &= x_k + \Delta x, \\ \lambda_{k+1} &= \lambda_k + \Delta\lambda, \end{aligned} \tag{3.3}$$

where the search direction $(\Delta x^T, \Delta\lambda)$ is

$$\begin{aligned} \Delta\lambda &= (\nabla_x h(x_k)^T (I + \lambda_k \mathcal{H}^*\mathcal{H})^{-1} \nabla_x h(x_k))^{-1} \{h(x_k) \\ &\qquad - \nabla_x h(x_k)^T (I + \lambda_k \mathcal{H}^*\mathcal{H})^{-1} \nabla_x L(x_k, \lambda_k)\}, \\ \Delta x &= -(I + \lambda_k \mathcal{H}^*\mathcal{H})^{-1} (\nabla_x L(x_k, \lambda_k) + \nabla_x h(x_k) \Delta\lambda). \end{aligned} \tag{3.4}$$

Proof. The Hessian $\nabla^2_{xx} L(x,\lambda)$ of the Lagrangian function $L(x,\lambda)$ is $I + \lambda\mathcal{H}^*\mathcal{H}$, which is also Toeplitz-plus-Hankel matrix with Toeplitz-plus-Hankel blocks. As usual a Taylor series for $\nabla_{x,\lambda} L(x_{k+1}, \lambda_{k+1})$ about (x_k^T, λ_k) gives

$$\begin{aligned} &\nabla_{x,\lambda} L(x_k + \Delta x, \lambda_k + \Delta\lambda) \\ &= \nabla_{x,\lambda} L(x_k, \lambda_k) + \nabla^2_{x,\lambda} L(x_k, \lambda_k) \begin{pmatrix} \Delta x \\ \Delta\lambda \end{pmatrix} \\ &= \nabla_{x,\lambda} L(x_k, \lambda_k) + \begin{pmatrix} I + \lambda_k \mathcal{H}^*\mathcal{H} & \nabla_x h(x_k) \\ \nabla_x h(x_k)^T & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda \end{pmatrix} + \cdots \end{aligned} \tag{3.5}$$

Neglecting higher order terms and setting the left hand size to zero gives the iteration

$$\begin{pmatrix} I + \lambda_k \mathcal{H}^*\mathcal{H} & \nabla_x h(x_k) \\ \nabla_x h(x_k)^T & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} \nabla_x L(x_k, \lambda_k) \\ h(x_k) \end{pmatrix}.$$

Thus the search direction $(\Delta x^T, \Delta\lambda)$ of (3.4) can be obtained by solving the above system directly. □

In the computation of (3.4), $\varpi = (I + \lambda\mathcal{H}^*\mathcal{H})^{-1}\upsilon$ is obtained as

$$\varpi = vector(extract(idct2(dct2(V_e)./(1 + \lambda|dct2(H_e)|.^\wedge 2)))),$$

where $|\cdot|$ and $.^\wedge$ are the component-wise absolute value and squaring of the matrix $dct2(H_e)$.

Note that the step-length α_k can be chosen by arbitrary line search method like the Armijo's condition. Including the step-length α_k along $(\Delta x^T, \Delta\lambda)$ to (3.3),

$$x_{k+1} = x_k + \alpha_k\Delta x, \lambda_{k+1} = \lambda_k + \alpha_k\Delta\lambda. \tag{3.6}$$

The stopping rule of the FTL method is based on the discrepancy principle. That is, the iteration (2.6) is terminated as soon as an iterate x_k has been found such that $\|\mathcal{H}x_k - y\| \le \rho\delta$, where $\rho \ge 1$ is a fixed parameter. Moreover the method is stopped if a maximum number Max of iterations has been reached without finding a solution satisfying the discrepancy principle([3, 11]).

The FTL algorithm for the image restoration problem with reflective boundary condition can be outlined as follows :

Algorithm 1. FTL algorithm for image restoration problem with reflective boundary condition.

1. Input $x_0, \lambda_0, y, \rho, \delta$, Max.
2. **For** $k = 0, 1, 2, \dots$.
   i. Compute
   $$\begin{aligned}\Delta\lambda &= (\nabla_x h(x_k)^T(I + \lambda_k\mathcal{H}^*\mathcal{H})^{-1}\nabla_x h(x_k))^{-1}\{h(x_k)\\ &\qquad - \nabla_x h(x_k)^T(I + \lambda_k\mathcal{H}^*\mathcal{H})^{-1}\nabla_x L(x_k, \lambda_k)\},\\ \Delta x &= -(I + \lambda_k\mathcal{H}^*\mathcal{H})^{-1}(\nabla_x L(x_k, \lambda_k) + \nabla_x h(x_k)\Delta\lambda).\end{aligned}$$
   ii. Find the step length α_k along $(\Delta x^T, \Delta\lambda)$.
   iii. $x_{k+1} = x_k + \alpha_k\Delta x$, $\lambda_{k+1} = \lambda_k + \alpha_k\Delta\lambda$.
   iv. If $\|\mathcal{H}x_{k+1} - y\| \le \rho\delta$ or $k \ge Max$, then stop.

4. Experimental results

We demonstrated the efficiency of employing reflective boundary condition over periodic boundary condition for image restoration problems.

To show how well the points approximate the true image, we investigated the relative accuracy, $\frac{\|x_{true} - x_{approx}\|}{\|x_{true}\|}$, and the PSNR(peak signal-to-noise ratio) values of the recovered images, $PSNR = 10\log 10\left(\frac{255^2}{MSE}\right)$.

Here MSE is the mean square error for two $m \times n$ monochrome images I and J, where one of the images is considered a noisy approximation of the other and is defined as $MSE(I,J) = \frac{\Sigma_{i,j}(I(i,j)-J(i,j))^2}{mn}$. PSNR is most commonly used as a measure of quality of reconstruction image([5]). As the relative accuracy gets smaller with the bigger values of PSNR, the approximated image becomes better.

The data source is a *moon* image taken from [2]. First, *Atmospheric turbulence* is due to random variations in the refractive index of the medium between the object and the imaging system. For many practical purposes, the blurring can be modeled by a Gaussian point spread function, $\mathcal{H}(x-x', y-y') = \frac{1}{2\pi\sigma\bar{\sigma}} \exp\left(-\frac{1}{2}\left(\frac{x-x'}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{y-y'}{\bar{\sigma}}\right)^2\right)$, where $\bar{\sigma}$ and σ are two constants that the blurring in the x and y directions, respectively. Second, *Out-of-focus blur* arises when the lens is out of focus, i.e., when the focal point of the lens does not match with the light-sensitive CCD. A good model of the point spread function for out-of focus blur is

$$\mathcal{H}(x-x', y-y') = \begin{cases} (\pi r^2)^{-1}, & \sqrt{(x-x')^2 + (y-y')^2} \leq r, \\ 0, & otherwise, \end{cases}$$

where the parameter r characterizes the defocus. These PSF functions have been obtained with the Matlab code *psfGauss* and *psfDefocus* from the *RestoreTools* package ([12]) respectively. In our test, we fix the parameter $\sigma = \bar{\sigma} = 5$ in the atmospheric turbulence and $r = 10$ in the out-of-focus blur.

Under the reflective boundary condition, the left of Fig. 1 shows the blurred and noisy image for comparison and the right of it is the reconstruction image by FTL scheme. Armijo's step-lengths are $\alpha_1 = \alpha_2 = \ldots = \alpha_{12} = 1$, $\alpha_{13} = \alpha_{14} = \frac{1}{2}$, $\alpha_{15} = \ldots = \alpha_{17} = \frac{1}{2^2}$, $\alpha_{18} = \alpha_{19} = \ldots = \alpha_{21} = \frac{1}{2^3}$, $\alpha_{22} = \alpha_{23} = \ldots = \alpha_{26} = \frac{1}{2^4}$, ... The discrepancy, $|\,\|\mathcal{H}x_{approx} - y\| - \delta|$, is 5.87×10^{-3}. The relative accuracy and PSNR corresponding Gaussian blurring for two boundary conditions are presented in Table 1. Using reflective boundary condition, the relative error and PSNR of reconstructed image are 3.79×10^{-2} and 37.77 respectively. Using the periodic boundary condition brings the result that the relative error is 1.04×10^{-1} and PSNR is 28.99. Relative accuracy is smaller and PSNR is higher under the reflective boundary condition. Also, we represented the result using the periodic boundary condition in Fig 2. There are a little ringing effect in the reconstruction image.

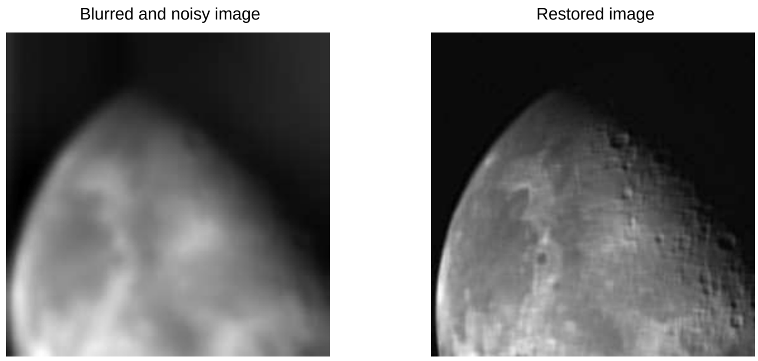

Figure 1. Restoring Gaussian blur with reflective boundary condition : Relative accuracy is 3.79×10^{-2}, PSNR is 37.77, and the discrepancy is 5.87×10^{-3}.

Thus reflective boundary condition brings a better result than periodic boundary condition.

We need two-dimensional FFTs and FCTs to compute the restored images for the periodic and the reflective boundary condition, respectively. The reflective boundary condition is twice as fast because FCTs require real multiplications only. In conclusion, we see that the cost using the reflective boundary condition is lower than that of using the periodic boundary condition.

Table 1. Comparison for reflective and periodic boundary condition under the Gaussian blurring

Boundary condition	Relative error	PSNR(dB)
Reflective	3.79×10^{-2}	37.77
Periodic	1.04×10^{-1}	28.99

Our results for out-of-focus blurring are presented in Table 2. Although different blurring, reflective boundary condition is more efficient than other boundary conditions.

Fig. 1 and 2 show the restored image for the two different boundary conditions respectively. By imposing the reflective boundary condition, the relative accuracy and the ringing effect are the smallest. *Moon* image is better reconstructed by using the reflective boundary condition than by using periodic boundary condition.

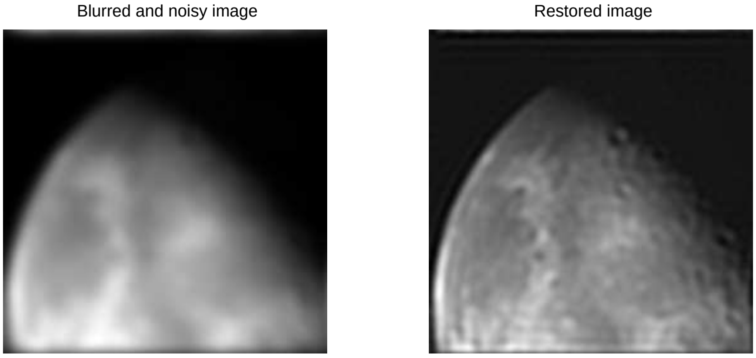

Figure 2. Restoring Gaussian blur with periodic boundary condition : Relative accuracy is 1.04×10^{-1}, PSNR is 28.99, and the discrepancy is 3.37×10^{-3}.

Table 2. Comparison for reflective and periodic boundary condition under the out-of-focus blurring

Boundary condition	Relative error	PSNR(dB)
Reflective	4.98×10^{-2}	35.39
Periodic	5.95×10^{-2}	33.86

5. Conclusion

Using reflective boundary condition in the image restoration problem brings a large-scale ill-conditioned Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks linear system. FTL method is also well suited for this system. Numerical results presented shows that the reflective boundary condition provides an effective model for image restoration problems, in terms of both the computational cost and minimizing the ringing effects near the boundary. The reflective boundary condition provides the smaller relative accuracy and the higher PSNR.

The artificial boundary discontinuities can be eliminated by using the reflective boundary condition. The choice of the most appropriate boundary conditions must be the one of the main aspect in the modelization of image deblurring. For a further research in the future, FTL algorithm can be extended to the anti-reflective and synthetic boundary condition to obtain a better result.

References

[1] A. Buades, B. Coll, and J. M. Morel, *Image Denoising Methods. A New Non-local Principle*, SIAM Review **52** (2010), no. 1, 113-147.

[2] R. C. Gonzalez and R. E. Woods, *Digital image processing,* Prentice Hall, 2002.

[3] G. Landi, *A fast truncated Lagrange method for large-scale image restoration problems,* Applied Mathematics and Computation **186** (2007), 1075-1082.

[4] A. K. Jain, *Fundamental of digital image processing*, Prentice-Hall, Engelwood Cliffs, NJ 1989.

[5] V. B. Surya Prasath and Arindama Singh, *A hybrid convex variational model for image restoration*, Applied Mathematics and Computation **215** (2010), 3655-3664.

[6] M. K. Ng, R. H. Chan, and W. C. Tang, *A fast algorithm for deblurring models with Neumann boundary conditions*, SIAM J. Sci. Comp. **21** (1999), no. 3, 851-866.

[7] M. K. Ng, W. C. Kwan, *Image restoration by cosine transform-based iterative regularization*, Applied Mathematics and Computation **160** (2005), 499-515.

[8] P. C. Hansen, *Regularization Tools,* (2001), http://www.imm.dtu.dk/ pch.

[9] P. C. Hansen, *Deconvolution and Regularization with Toeplitz matrices,* Numerical Algorithms, **29** (2002), 323-378.

[10] P. C. Hansen, J. G. Nagy, and D. P. O'Leary, *Deblurring Images Matrices, Spectra, and Filtering*, SIAM, 2006.

[11] M. E. Kilmer and D. P. O'Leary, *Choosing Regularization Parameters in Iterative Methods for Ill-Posed Problems,* SIAM J. Matrix Anal. Appl. **22** (2001), 1204-1221.

[12] K. P. Lee, J. G. Nagy, and L. Perrone, *Iterative Methods for Image Restoration: A Matlab Object Oriented Approach,* Numerical Algorithms **36** (2004), 73-93.

[13] S. Serra-Cappizzano, *A note on antireflectibe boundary conditions and fast deblurring models*, SIAM J. Sci. Comp. **25** (1999), no. 4, 851-866.

*

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: `soh@cnu.ac.kr`

**

Innovation Center of Engineering Education
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: `sjkw@cnu.ac.kr`